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Composition operators with weak hyponormality

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Abstract

There are many operator classes that are weaker than p -hyponormal. These include p -quasihyponormal, absolute p -paranormal, p -paranormal, normaloid, and spectraloid. In this note, we discuss measure theoretic composition operators in these classes.

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1. Introduction

This note is a continuation of the work done in [2]. In that work, Alan Lambert and the authors of this article determined when measure theoretic composition operators were p -hyponormal, ∞ -hyponormal, and w -hyponormal. Definitions of these classes will be given below. (Characterizations for normal composition operators, quasinormal composition operators, and subnormal composition operators were previously known.) In [2], examples were given which show that composition operators can be used to separate each partial normality class from quasinormal through w -hyponormal. We now turn our attention to classes that are weaker than p -hyponormal such as p -quasihyponormal, absolute p -paranormal, p -paranormal, normaloid, and spectraloid. Here is a brief review of what characterizes membership in each class. (See [4] for further discussion.)

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Let \mathcal{H} be the infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on \mathcal{H} . Let $A = U|A|$ be the canonical polar decomposition for $A \in \mathcal{L}(\mathcal{H})$ and let $p \in (0, \infty)$. An operator A is *p-hyponormal* if $(A^*A)^p \geq (AA^*)^p$. And A is *p-quasihyponormal* if $A^*(A^*A)^p A \geq A^*(AA^*)^p A$. For all unit vectors $x \in \mathcal{H}$, if $\| |A|^p U |A|^p x \| \geq \| |A|^p x \|^2$, then A is called a *p-paranormal operator*. In particular, 1-paranormal is referred to as paranormal. An operator A is of *A(p)-class* if $(A^*|A|^{2p}A)^{1/(p+1)} \geq |A|^2$, and *absolute-p-paranormal operator* if $\| |A|^p Ax \| \geq \| Ax \|^{p+1}$ for all unit vectors x in \mathcal{H} . Note that absolute-1-paranormal is the same as 1-paranormal. Let $\tilde{A} := |A|^{1/2} U |A|^{1/2}$ be the *Aluthge transform* of A . Then A is *w-hyponormal* if $|\tilde{A}| \geq |A|$ [1,6]. An operator A is *normaloid* if $\|A\| = r(A)$, where $r(A)$ is the spectral radius of A , which is equivalent to the condition $\|A^n\| = \|A\|^n$ for all natural numbers n (see [4, p. 100]). An operator A is *spectraloid* if $w(A) = r(A)$, where $w(A)$ is the numerical radius of A .

There are several well-known relationships among these classes [4]. The ones of concern in this note are as follows: p -hyponormal \Rightarrow p -quasihyponormal \Rightarrow $A(p)$ -class operator \Rightarrow absolute- p -paranormal \Rightarrow normaloid \Rightarrow spectraloid ($p > 0$); absolute- p -paranormal \Rightarrow p -paranormal ($p \geq 1$); p -paranormal \Rightarrow absolute- p -paranormal ($0 < p < 1$); w -hyponormal \Rightarrow $\frac{1}{2}$ -paranormal (Example 3.1 shows this implication cannot be strengthened). For $0 < p < q$, if T is p -paranormal, then A is q -paranormal. All the other p -properties except p -hyponormality share this type of implication. For p -hyponormality, the implication is reversed: if A is q -hyponormal, then A is p -hyponormal.

In this article, we show that composition operators can separate the w -hyponormal, p -paranormal, normaloid, and spectraloid classes. They cannot, however, be used to separate the p -quasihyponormal, $A(p)$ -class, absolute p -paranormal, or p -paranormal classes (see Theorem 2.3).

Before giving our results, we briefly review some essential notation and background information on composition operators. Let (X, \mathcal{F}, μ) be a σ finite measure space and let $T: X \rightarrow X$ be a transformation such that $T^{-1}\mathcal{F} \subset \mathcal{F}$ and $\mu \circ T^{-1} \ll \mu$. We assume that the Radon–Nikodym derivative $h = d\mu \circ T^{-1}/d\mu$ is in L^∞ and we define $h_n = d\mu \circ T^{-n}/d\mu$. The composition operator C acting on $L^2 := L^2(X, \mathcal{F}, \mu)$ is defined by $Cf = f \circ T$. The condition $h \in L^\infty$ assures that C is bounded. We denote the conditional expectation of f with respect to $T^{-1}\mathcal{F}$ by $Ef = E(f | T^{-1}\mathcal{F})$. We recall some known results from [2,7], and [5], which will be used frequently through this paper. Every $T^{-1}\mathcal{F}$ measurable function has the form $F \circ T$ (hence Ef is of this form). Note that $F \circ T = G \circ T$ if and only if $hF = hG$; in fact, $F \circ T \geq G \circ T$ if and only if $F\chi_S \geq G\chi_S$ where $S = \text{support } h$ and χ_S is the characteristic function of S [3]. It is known that $C^*f = h(Ef) \circ T^{-1}$ (the previous two properties show that this expression is well defined) and $h \circ T > 0$ a.e.

In the proofs and examples that follow, we will need certain properties of the conditional expectation operator $E: E = E(\cdot | T^{-1}\mathcal{F})$ is the self adjoint projection onto $L^2(X, T^{-1}\mathcal{F}, \mu)$. For any $T^{-1}\mathcal{F}$ set A and L^2 function f , $\int_A f d\mu = \int_A E(f) d\mu$. For $T^{-1}\mathcal{F}$ measurable a and \mathcal{F} measurable f , $E(af) = aE(f)$. The interested reader can find a more extensive list of properties for conditional expectations in [2].

2. Characterizations

In this section we determine necessary and sufficient conditions for a composition operator to be p -quasihyponormal, an $A(p)$ -class operator, absolute p -paranormal, or p -paranormal. A characterization of normaloid operators in terms of the Radon–Nikodym derivatives $h_n, n =$

1, 2, ..., is given in the remark at the end of this section. The characterizations for the classes above are transparent: they only depend on h (or h_n), T , and the conditional expectation E . We are unable to characterize spectraloid composition operators in this same fashion. However, in Example 3.1 we show that there are spectraloid composition operators which are not normaloid.

We begin with the following lemmas:

Lemma 2.1. *Let $A \in \mathcal{L}(\mathcal{H})$ and let $U|A|$ be its polar decomposition. Suppose $p \in (0, \infty)$. Then we have the following:*

(i) [4, p. 174] *A is absolute- p -paranormal if and only if*

$$A^*|A|^{2p}A - (p+1)\lambda^p|A|^2 + p\lambda^{p+1} \geq 0, \quad \text{for all } \lambda \geq 0. \quad (2.1)$$

(ii) [8, Proposition 3] *A is p -paranormal if and only if*

$$|A|^p U^*|A|^{2p} U|A|^p - 2\lambda|A|^{2p} + \lambda^2 \geq 0, \quad \text{for all } \lambda \geq 0. \quad (2.2)$$

Lemma 2.2. *Let C be a composition operator on L^2 . Then C is p -quasihyponormal if and only if $E(h^p) \geq h^p \circ T$.*

Proof. By a simple computation, we have

$$C^*(C^*C)^p C f = h(E(h^p) \circ T^{-1})f, \quad f \in L^2,$$

and, because E commutes with multiplication by $h \circ T$ so that $(CC^*)^p f = (h \circ T)^p E f$,

$$C^*(CC^*)^p C f = h^{p+1} f, \quad f \in L^2.$$

Both of these operators are multiplication operators, hence $C^*(C^*C)^p C \geq C^*(CC^*)^p C$ if and only if $h(E(h^p) \circ T^{-1}) \geq h^{p+1}$. Composing with T and using the fact that $h \circ T > 0$ this is equivalent to $E(h^p) \geq h^p \circ T$. \square

Theorem 2.3. *Let C be a composition operator on L^2 . Then the following are equivalent:*

- (i) C is p -quasihyponormal;
- (ii) C is an $A(p)$ -class operator;
- (iii) C is absolute- p -paranormal;
- (iv) C is p -paranormal;
- (v) $E(h^p) \geq h^p \circ T$.

Proof. Note that (v) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) by Lemma 2.2 and the general implications given in the introduction. We complete the proof by showing (iii) \Leftrightarrow (v) and (iv) \Leftrightarrow (v) below.

We first prove (iii) \Leftrightarrow (v). Since $C^*|C|^{2p}C f = hE(h^p) \circ T^{-1} f$ and $|C|^2 f = h f$ are multiplication operators, condition (2.1) for absolute- p -paranormality of C is equivalent to

$$H(\lambda) := hE(h^p) \circ T^{-1} - (p+1)\lambda^p h + p\lambda^{p+1} \geq 0 \quad \text{a.e., for all } \lambda \geq 0.$$

Note that $H(\lambda)$ is of the form $a - b(p+1)\lambda^p + p\lambda^{p+1}$ with $a, b \geq 0$. This function has a minimum on $[0, \infty)$ at $\hat{\lambda} = b$. Therefore

$$H(\hat{\lambda}) \geq 0 \quad \Leftrightarrow \quad hE(h^p) \circ T^{-1} \geq h^{p+1} \quad \Leftrightarrow \quad E(h^p) \geq h^p \circ T,$$

which proves the desired implication.

We now prove (iv) \Leftrightarrow (v). Since $Uf = (h \circ T)^{-1/2} f \circ T$ and $U^*f = h^{1/2} E(f) \circ T^{-1}$, by a direct computation, we have

$$|C|^p U^* |C|^{2p} U |C|^p f = h^p [E(h^p) \circ T^{-1}] f, \quad f \in L^2.$$

Because this is a multiplication operator, condition (2.2) for p -paranormality of C is equivalent to

$$F(\lambda) := h^p [E(h^p) \circ T^{-1}] - 2\lambda h^p + \lambda^2 \geq 0 \quad \text{a.e.}$$

This quadratic expression is a minimum when $\lambda = h^p$, hence

$$F(\lambda) \geq 0 \quad (\lambda \geq 0) \quad \Leftrightarrow \quad h^p E(h^p) \circ T^{-1} \geq h^{2p} \quad \Leftrightarrow \quad E(h^p) \geq h^p \circ T.$$

Hence the proof is complete. \square

Remark. Notice that, since $\|C\| = \|h\|_\infty^{1/2}$ and $\|C^n\| = \|h_n\|_\infty^{1/2}$ (recall that $h_n = d\mu \circ T^{-n}/d\mu$), C is normaloid if and only if $\|h\|_\infty = \|h_n\|_\infty^{1/n}$ for all $n \in \mathbb{N}$. We use this characterization in Examples 3.2 and 3.5.

3. Examples

In Examples 3.1–3.3 we use a 2-parameter family of composition operators to separate the w -hyponormal, p -paranormal, normaloid, and spectraloid classes. This same family was used in [2, Example 3.1] to show that composition operators can separate all p -hyponormality classes.

Example 3.1 (Paranormality). Let X be the set of nonnegative integers, let \mathcal{F} be the σ algebra of all subsets of X , and take μ to be the measure determined by the strictly positive sequence $\{m_k\}_{k \geq 0}$ given below. Our point transformation T is defined by

$$T(k) = \begin{cases} 0, & k = 0, 1, 2, \\ k-2, & k \geq 3. \end{cases}$$

The action of T may be viewed as two paths leading back to 0, with 0 tied to itself. Note that the σ algebra $T^{-1}\mathcal{F}$ is generated by the atoms $\{0, 1, 2\}$, $\{3\}$, $\{4\}$, \dots . We specify our point mass measure m as follows (initializing the sequence at m_0):

$$m = 1, 1, 1, c, d, c^2, d^2, c^3, d^3, \dots,$$

where c and d are fixed positive numbers. The powers of c occur for odd integers and those of d for even integers. A simple calculation shows that, as a sequence, $h = 3, c, d, c, d, \dots$ and consequently $h \circ T = 3, 3, 3, c, d, c, \dots$. Furthermore,

$$Ef = \frac{f_0 + f_1 + f_2}{3}, \frac{f_0 + f_1 + f_2}{3}, \frac{f_0 + f_1 + f_2}{3}, f_3, f_4, \dots$$

Now fix a number $p > 0$. Then

$$E(h^p) = \frac{3^p + c^p + d^p}{3}, \frac{3^p + c^p + d^p}{3}, \frac{3^p + c^p + d^p}{3}, c^p, d^p, c^p, \dots$$

By Theorem 2.3, C is p -paranormal if and only if $(\frac{c}{3})^p + (\frac{d}{3})^p \geq 2$. After some computations, one can show

$$\bigcap_{p>0} \{(c, d) \mid C \text{ is } p\text{-paranormal}\} = \{(c, d) \mid cd \geq 9\}$$

and

$$\bigcup_{p>0} \{(c, d) \mid C \text{ is } p\text{-paranormal}\} = \{(c, d) \mid c > 3 \text{ or } d > 3\} \cup \{(3, 3)\}.$$

Using the characterization of p -hyponormality in [2] we also have

$$\begin{aligned} \bigcup_{p>0} \{(c, d) \mid C \text{ is } p\text{-hyponormal}\} &= \bigcup_{p>0} \left\{ (c, d) \mid \left(\frac{3}{c}\right)^p + \left(\frac{3}{d}\right)^p \leq 2 \right\} \\ &= \{(c, d) \mid cd > 9\} \cup \{(3, 3)\}. \end{aligned}$$

We now show that composition operators can separate all p -paranormality classes. Fix $p > 0$ and choose any d such that $3 < d < 3(2^{1/p})$. Then find $c > 0$ such that $(c/3)^p + (d/3)^p = 2$. Then C is p -paranormal. Let $0 < q < p$. We will show that C is not q -paranormal. With $A = c/3$, $B = d/3$, and $f: [0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = A^x + B^x$, C is q -paranormal if and only if $f(q) \geq 2$. But $f(0) = f(p) = 2$, and $f''(x) > 0$ for all x , implies $f(q) < 2$. Thus, C is not q -paranormal.

In [2], we found that the composition operator in this example is w -hyponormal if and only if $(c/3)^{1/2} + (d/3)^{1/2} \geq 2$, i.e. if and only if C is $1/2$ -paranormal. The discussion above shows that there are w -hyponormal composition operators which are not q -hyponormal for any $q \in (0, 1/2)$. This proves that the general implication w -hyponormal $\Rightarrow (1/2)$ -paranormal given in the introduction cannot be improved.

Example 3.2 (Normaloid). Using the family of composition operators given in Example 3.1, we now determine when C is normaloid. We have

$$\begin{aligned} h_2: & \quad 3 + (c + d), c^2, d^2, c^2, d^2, \dots, \\ h_3: & \quad 3 + (c + d) + (c^2 + d^2), c^3, d^3, c^3, d^3, \dots, \\ & \quad \vdots \\ h_n: & \quad 1 + \frac{c^n - 1}{c - 1} + \frac{d^n - 1}{d - 1}, c^n, d^n, c^n, d^n, \dots, \\ & \quad \vdots \\ & \quad \text{etc.} \end{aligned}$$

If $0 < c, d < 3$, then $\|h\|_\infty = 3$. Since $\|h_2\|_\infty < 9$, C cannot be normaloid. We now assume that $c \geq d$ and $c \geq 3$. Then $\|h\|_\infty = c$. Because

$$h_n(0) = 1 + \frac{c^n - 1}{c - 1} + \frac{d^n - 1}{d - 1} \leq 1 + 2 \frac{c^n - 1}{c - 1} \leq c^n, \quad \text{for } n = 0, 1, 2, \dots,$$

we have $\|h_n\|_\infty^{1/n} = c = \|h\|_\infty$, for all $n \in \mathbb{N}$. Thus, C is normaloid. Similarly, C is normaloid if $d \geq c$ and $d \geq 3$. Consequently, C is normaloid but not p -paranormal for any $p > 0$ if and only if (c, d) is in the set $\{(3, d) \mid 0 \leq d < 3\} \cup \{(c, 3) \mid 0 \leq c < 3\}$. Thus, composition operators can separate the normaloid and p -paranormal classes. Of course, this also separates the normaloid and w -hyponormal classes (w -hyponormal $\Rightarrow 1/2$ -paranormal).

Example 3.3 (Spectraloid). Finally, we show that there is a region where our family of composition operators is spectraloid, but not normaloid. Because of the discussion above, we restrict our attention to the region $0 < c, d < 3$. Without loss of generality, we assume that $c \geq d$. We will show that C is spectraloid when $c \geq (1 + \sqrt{5})^2/4 \approx 2.618$. We also show that when $d \leq c < 2.249$, C is not spectraloid. The explicit formula for h_n given in Example 3.2 can be used to show that $r(C) = \lim_{n \rightarrow \infty} \|h_n\|_\infty^{1/(2n)} = \max\{1, \sqrt{c}\}$. Since the inequality $w(C) \geq r(C)$ always holds, C will be spectraloid if we can show $w(C) \leq \max\{1, \sqrt{c}\}$. To this end, suppose that, in vector form, $f = f_0, f_1, f_2, \dots$. Then with

$$a_k = \begin{cases} |f_0| & \text{if } k = 0, \\ c^{(k-1)/4} |f_k| & \text{if } k = 1, 3, 5, \dots, \\ d^{(k-1)/4} |f_k|, & \text{if } k = 2, 4, 6, \dots, \end{cases}$$

we have $\|f\|^2 = \sum_{k=0}^{\infty} |a_k|^2$ and (with $A = a_0^2 + a_0 a_1 + a_0 a_2$)

$$\begin{aligned} |\langle Cf, f \rangle| &\leq A + \sqrt{c} \sum_{k=1}^{\infty} a_{2k-1} a_{2k+1} + \sqrt{d} \sum_{k=1}^{\infty} a_{2k} a_{2k+2} \\ &\leq A + \sqrt{c}/2 \sum_{k=1}^{\infty} (a_{2k-1}^2 + a_{2k+1}^2) + \sqrt{d}/2 \sum_{k=1}^{\infty} (a_{2k}^2 + a_{2k+2}^2) \\ &= A + (\sqrt{c}/2) a_1^2 + (\sqrt{d}/2) a_2^2 + \sqrt{c} \sum_{k=1}^{\infty} a_{2k+1}^2 + \sqrt{d} \sum_{k=1}^{\infty} a_{2k+2}^2 \\ &\leq A + (\sqrt{c}/2) a_1^2 + (\sqrt{c}/2) a_2^2 + \sqrt{c} \sum_{k=1}^{\infty} a_{2k+1}^2 + \sqrt{c} \sum_{k=1}^{\infty} a_{2k+2}^2 \\ &= A + (\sqrt{c}/2) (a_1^2 + a_2^2) + \sqrt{c} (\|f\|^2 - a_0^2 - a_1^2 - a_2^2) \\ &= (1 - \sqrt{c}) a_0^2 + a_0 a_1 + a_0 a_2 - (\sqrt{c}/2) (a_1^2 + a_2^2) + \sqrt{c} \end{aligned}$$

whenever $\|f\| = 1$. We conclude that

$$\begin{aligned} w(C) &= \sup\{|\langle Cf, f \rangle| \mid \|f\| = 1\} \\ &\leq (1 - \sqrt{c}) a_0^2 + a_0 a_1 + a_0 a_2 - (\sqrt{c}/2) (a_1^2 + a_2^2) + \sqrt{c}. \end{aligned}$$

Assume $c \geq 1$. Set $a = \sqrt{c}$ and $k = \sqrt{a-1}$. Then, rewriting the right-hand side of the inequality above, we have

$$w(C) \leq -\frac{1}{2} \left(ka_0 - \frac{a_1}{k} \right)^2 - \frac{1}{2} \left(ka_0 - \frac{a_2}{k} \right)^2 + \frac{1}{2} \left(\frac{1}{k^2} - a \right) (a_1^2 + a_2^2) + a.$$

We will have $w(C) \leq a = \sqrt{c}$ whenever $k^{-2} - a \leq 0$, i.e. when $a \geq (1 + \sqrt{5})/2$. This proves C is spectraloid when $c \geq d$ and $\sqrt{c} \geq (1 + \sqrt{5})/2$, i.e. when $c > 2.618$. We conclude that if $c \geq d$ and $2.618 < c < 3$, then C is spectraloid, but not normaloid.

We have already accomplished our goal of showing that composition operators can separate the spectraloid and normaloid classes, but unfortunately, we are currently unable to fully determine the region $\{(c, d) \mid C \text{ is spectraloid}\}$. However, we are able to limit this region: We first prove that C is not spectraloid when $0 < d \leq c < (1 + \sqrt{3})^2/4$. Let $0 \leq x \leq 1/\sqrt{2}$ and set $f_1 = f_2 = x$, $f_0 = \sqrt{1 - 2x^2}$, and $f_n = 0$ for $n \geq 3$. Then f has norm 1 and $\langle Cf, f \rangle =$

$1 - 2x^2 + 2x\sqrt{1 - 2x^2}$. Maximizing this function over the region $0 \leq x \leq 1/\sqrt{2}$, we find that $x = (1/2)\sqrt{1 - 1/\sqrt{3}}$ gives a maximum value of $(1 + \sqrt{3})/2$. This proves that $w(C) \geq (1 + \sqrt{3})/2$, but $r(C) = \max\{1, \sqrt{c}\}$ (see above) so that C is not spectraloid if $(1 + \sqrt{3})^2/4 > c$.

We now improve the estimate for the region where C is not spectraloid. The result obtained above allows us to restrict our attention to the case $c \geq d$ and $c > 1$. This assures that $r(C) = \sqrt{c}$. If we can demonstrate $c_0 > 1$ such that $C = C_{c_0d}$ is not spectraloid, then C_{cd} will not be spectraloid for $1 < c < c_0$: Define $g(c, d, f) := \langle C_{cd}f, |f| \rangle - r(C) = \langle C_{cd}f, |f| \rangle - \sqrt{c}$. The operator C_{c_0d} is not spectraloid if and only if there is a unit vector f such that $g(c_0, d, f) > 0$. Then, with this f and $c_0 > c > 1$ (notation as above),

$$\begin{aligned} g(c_0, d, f) - g(c, d, f) &= (\sqrt{c_0} - \sqrt{c}) \left(\sum_{k=1}^{\infty} a_{2k-1} a_{2k+1} - 1 \right) \\ &\leq (\sqrt{c_0} - \sqrt{c}) \left(\frac{1}{2} a_1^2 + \sum_{k=1}^{\infty} a_{2k+1}^2 - 1 \right) \\ &\leq (\sqrt{c_0} - \sqrt{c}) (\|f\|^2 - 1) \\ &\leq 0. \end{aligned}$$

Thus, $0 < g(c_0, d, f) \leq g(c, d, f)$ and C_{cd} is also nonspectraloid.

We now demonstrate c_0 and a unit vector f such that $g(c_0, d, f) > 0$, i.e. C_{c_0d} is not spectraloid. Let $c = c_0 = 2.249$ and let $0 < r < 1$. Setting $a_4 = a_6 = a_8 = \dots = 0$ and $a_{2n+3} = r^n a_3$ for $n \in \mathbb{N}$, we have

$$\langle Cf, f \rangle = a_0^2 + a_0 a_1 + a_0 a_2 + \sqrt{c} a_1 a_3 + \frac{r \sqrt{c} a_3^2}{1 - r^2}$$

and $\|f\|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2/1 - r^2$. Defining $\hat{a}_3 = a_3/\sqrt{1 - r^2}$, fixing $r = 0.999$, and using the method of Lagrange multipliers, we find that $a_0 = 0.06780$, $a_1 = 0.04493$, $a_2 = 0.02263$, $\hat{a}_3 = 0.9964297$ gives a unit vector with $g(c_0, d, f) \approx 8.44 \times 10^{-7} > 0$.

However, we do not know the exact function $f(c, d) = 0$ for the boundary of the region $\{(c, d) \mid C \text{ is not spectraloid}\}$.

Example 3.4 (Graph). Putting our results together, we have Fig. 1 which clearly illustrates the separation of the weak hyponormality classes discussed above.

Example 3.5. (An example using a completely nonatomic measure space.) The family of operators given above was based on a purely atomic measure space. To dispel any notion that measure theoretic composition operators provide useful examples only in this setting, we now provide a second example of a normaloid composition operator that is not p -paranormal for any $p > 0$. Let $X = \mathbb{R}$ and let μ be Lebesgue measure. The transformation T is piecewise linear

$$T(x) = \begin{cases} x - 1 & \text{if } x \leq 1, \\ 2x - 2 & \text{if } 1 < x \leq 1.5, \\ 4 - 2x & \text{if } 1.5 < x \leq 2, \\ 2x - 2.5 & \text{if } x > 2. \end{cases}$$

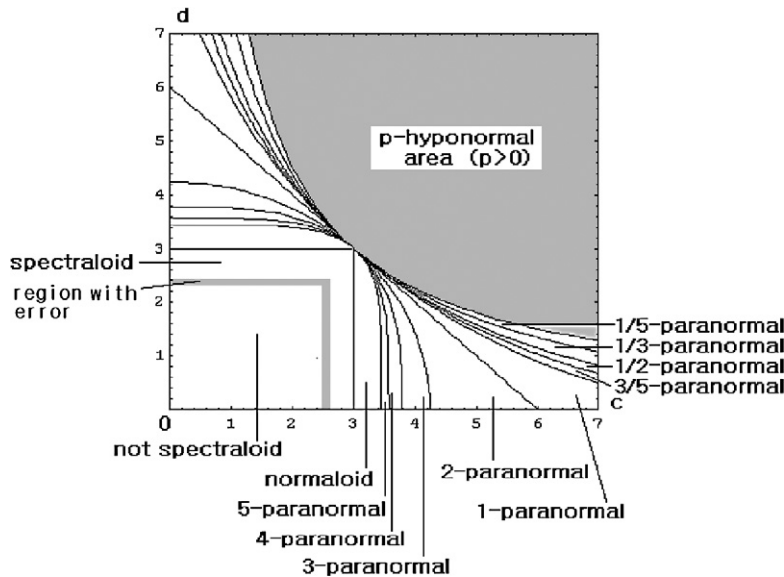


Fig. 1.

The σ algebra $T^{-1}\mathcal{F}$ consists of all Lebesgue measurable subsets of $(-\infty, 1) \cup (2, \infty)$, together with all Lebesgue measurable subsets of $(1, 2)$ that are symmetric about 1.5. Let $p > 0$ and let χ_A be the characteristic function of the set A . A straightforward computation shows

$$\begin{aligned} h &= \chi_{(-\infty, 1)} + \frac{1}{2}\chi_{(1.5, \infty)}, \\ h^p \circ T &= \chi_{(-\infty, 2)} + 2^{-p}\chi_{(2, \infty)}, \\ Eh^p &= \chi_{(-\infty, 1)} + 2^{-p-1}\chi_{(1, 2)} + 2^{-p}\chi_{(2, \infty)}. \end{aligned}$$

Since $Eh^p < h^p \circ T$ on the set $(1, 2)$, we see that C is not p -paranormal. However, $h_1 = h$, and, for $n \geq 2$ we have

$$\begin{aligned} h_n &= 2^{-n}\chi_{(1.5, \infty)} + 2^{-n}\chi_{(0, 1]} + 2^{1-n}\chi_{(-1, 0]} + 2^{2-n}\chi_{(-2, -1]} + \cdots \\ &\quad + 2^{-2}\chi_{(2-n, 3-n]} + \chi_{(-\infty, 2-n]}. \end{aligned}$$

We conclude that $\|h\|_\infty = \|h_n\|_\infty^{1/n}$ for all $n \in \mathbb{N}$. Consequently, C is normaloid.

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